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Brooks–Jewett and Nikodym Convergence Theorems for Orthoalgebras That Have the Weak Subsequential Interpolation Property

Eissa D. Habil[1]

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A weak form of σ-orthocompleteness or σ-orthosummability, referred to as the Weak Subsequential Interpolation Property, is defined for orthoalgebras. It is shown that the class of orthoalgebras that have this property properly contains the class of σ-orthoalgebras. The Brooks–Jewett theorem and the Nikodym convergence theorem for semigroup-valued finitely additive and s-bounded measures defined on an orthoalgebra satisfying the Weak Subsequential Interpolation Property are proved.

1. INTRODUCTION

According to Gudder (1988), quantum mechanics is a probabilistic theory, and a complete description of a quantum mechanical system is given by a probability measure on its set of events. This set of events fails to form a σ-field (or a σ-complete Boolean algebra), an algebraic structure that provides a foundation for classical measure theory. It rather forms a σ-complete orthomodular lattice, an algebraic structure less rich than a σ-field. This has given birth to the part on noncommutative measure theory that deals with the study of measures and states on non-Boolean orthostructures such as orthomodular lattices and posets (Birkhoff and von Neumann, 1936; Gleason, 1957; Mackey, 1963; Gudder, 1965, 1988; Greechie, 1971; Cook, 1978; D'Andrea and De Lucia, 1991; D'Andrea *et al.*, 1991; De Lucia and Morales, 1988; Rüttimann and Schindler, 1989; Navara and Rüttimann, 1991). The nonexistence of a tensor product for orthomodular lattices or posets has led to the study of orthoalgebras, a more general orthostructure a large class of which admits a

[1]Department of Mathematics and Computer Science, University of Denver, Denver, Colorado 80208.

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tensor product (Foulis *et al.*, 1992; Foulis and Bennett, 1993). Orthoalgebras (resp., σ-orthoalgebras) are apparently the simplest and most natural orthostructures that can carry orthogonally additive (resp., σ-additive) measures and thus are basic for the developing field of noncommutative measure theory (Rüttimann, 1979, 1989; Dvurečenskij and Riečan, 1994; Habil, 1994*a*,*b*; Younce, 1987).

In this paper, we introduce the *Weak Subsequential Interpolation Property* and the *Weak Subsequential Completeness Property* for orthoalgebras. Then we prove noncommutative versions of some important theorems from classical measure theory; namely, we prove a Brooks–Jewett theorem (Brooks and Jewett, 1970), a Nikodym convergence theorem (De Lucia and Morales, 1988; Diestel and Uhl, 1977), and a Cafiero uniform boundedness theorem (Cafiero, 1952) for orthoalgebras that satisfy these properties. Commutative versions of these selected theorems have been proven for σ-complete Boolean rings by Weber (1986), for Boolean rings that satisfy the Subsequential Interpolation Property by De Lucia and Morales (1988), and for Boolean algebras that satisfy the Subsequential Interpolation Property by Freniche (1984); noncommutative versions have been proven for σ-orthocomplete orthomodular posets by Morales (1988) and for orthomodular lattices that satisfy the Subsequential Interpolation Property by D'Andrea and De Lucia (1991).

We show that the class of orthoalgebras having the Weak Subsequential Interpolation (resp., Completeness) Property contains the class of orthomodular lattices having the Subsequential Interpolation (resp., Completeness) Property as defined in D'Andrea and De Lucia (1991). Hence we obtain, as an immediate consequence of the above-mentioned theorems, their counterparts, which have been established by D'Andrea and De Lucia (1991), for such orthomodular lattices. We also show that the class of orthoalgebras that have the Weak Subsequential Interpolation Property contains the class of σ-orthoalgebras, and thereby get as an immediate consequence of the above-mentioned theorems new versions for σ-orthoalgebras. We finally prove a Nikodym–Vitali–Hahn–Saks theorem (Cook, 1978; Dunford and Schwartz, 1957) for σ-orthoalgebras.

Our presentation is modeled along the lines of that of D'Andrea and De Lucia (1991). The proof of the Brooks–Jewett theorem is obtained by reducing it to the commutative setting of studying functions on rings of sets, and the proof of the Nikodym–Vitali–Hahn–Saks theorem is obtained by reducing it to the commutative setting of studying functions on σ-complete Boolean algebras.

Throughout this paper, the symbols $\mathcal{P}(X)$, $\mathcal{F}(X)$, $c\mathcal{F}(X)$, and $\mathcal{I}(X)$ denote, respectively, the collections of all subsets, all finite subsets, all cofinite subsets, and all infinite subsets of a set X. The symbols $\mathbb{R}$, $\mathbb{Z}$, and ω denote, respectively, the sets of all real numbers, all integers, and all nonnegative integers. The notation $:=$ means "equals by definition."

2. ORTHOALGEBRAS HAVING THE WEAK SUBSEQUENTIAL INTERPOLATION PROPERTY

Definition 2.1. An *orthoalgebra* (OA) is a quadruple $(L, \oplus, 0, 1)$ where L is a set containing two special elements 0, 1 and $\oplus$ is a partially defined binary operation on L that satisfies the following conditions $\forall p, q, r \in L$:

(OA1) (*Commutativity*) If $p \oplus q$ is defined, then $q \oplus p$ is defined and $p \oplus q = q \oplus p$.

(OA2) (*Associativity*) If $q \oplus r$ is defined and $p \oplus (q \oplus r)$ is defined, then $p \oplus q$ is defined, $(p \oplus q) \oplus r$ is defined, and $p \oplus (q \oplus r) = (p \oplus q) \oplus r$.

(OA3) (*Orthocomplementation*) For every $p \in L$ there exists a unique $q \in L$ such that $p \oplus q$ is defined and $p \oplus q = 1$.

(OA4) (*Consistency*) If $p \oplus p$ is defined, then $p = 0$.

We shall write L for the OA $(L, \oplus, 0, 1)$. Let L be an OA and $p, q \in L$. We say p is *orthogonal* to q in L and we write $p \perp q$ if and only if $p \oplus q$ is defined in L. We define $p \leq q$ to mean that there exists $r \in L$ such that $p \perp r$ and $q = p \oplus r$. The unique element q corresponding to p in condition (OA3) above is called the *orthocomplement* of p and is written as p'. It can be easily proved (Foulis *et al.*, 1992) that $p \perp q$ iff $p \leq q'$, that $0 \leq p \leq 1$ holds for all $p \in L$, that "$\leq$" as defined above is a partial ordering on L, that $(L, \leq, ', 0, 1)$ is an orthoposet, and that that for all $p, q \in L$,

$$p \leq q \Rightarrow q = p \oplus (p \oplus q')'$$

the so-called *orthomodular identity* (OMI). For $p, q \in L$, p is called a *subelement* of q iff $p \leq q$. If p is a subelement of q, then, by the OMI, $q = p \oplus (p \oplus q')'$. In this case we define the *difference* of q and p in L by

$$q - p := (p \oplus q')'$$

An *orthomodular poset* (OMP) is an orthoalgebra P that satisfies the following condition:

$$p, q \in L, \quad p \perp q \quad \Rightarrow \quad p \vee q \text{ exists and } p \vee q = p \oplus q$$

where $p \vee q$ denotes the least upper bound of $\{p, q\}$ in L. A σ-*orthocomplete* OMP is an OMP P in which every countable subset of P has a least upper bound. An *orthomodular lattice* (OML) is an OMP which is also a lattice. A *Boolean algebra* is a *distributive* OML.

Let L be an OA. A subset $A \subseteq L$ is called a *suborthoalgebra* (sub-OA) if $0, 1 \in A$, $p' \in A$ whenever $p \in A$, and $p \oplus q \in A$ whenever $p, q \in A$

and $p \perp q$. A sub-OA of an OA is, of course, an OA in its own right. If A is a sub-OA of L, then for $p, q \in A$ the notation $p \vee^A q$ (resp., $p \wedge^A q$) stands for the least upper bound (resp., the greatest lower bound) of $\{p, q\}$ as calculated in A.

Definition 2.2. Let L be an OA and $A \subseteq L$ be a sub-OA. Then A is called:

(i) A *sub-OMP* if $p, q \in A, p \perp q \Rightarrow p \vee^A q$ exists.
(ii) A *sub-OML* if $p, q \in A \Rightarrow p \vee^A q$ exists.
(iii) A *Boolean subalgebra* if it is a distributive sub-OML.
(iv) A *block* if it is a maximal Boolean subalgebra under set-theoretic inclusion.

A subset X of an OA L is called *jointly orthogonal* iff X is pairwise orthogonal and is contained in a block B of L. In the sequel, we shall use the notation

$$J(L) := \{X \subseteq L: X \text{ is jointly orthogonal}\}$$

Definition 2.3. (i) An OML L is called a SIP-OML iff it satisfies the *Subsequential Interpolation Property*: For every orthogonal sequence $(a_i)_{i\in\omega} \subseteq L$ and for every $N \in \mathcal{I}(\omega)$, there exist $M \in \mathcal{I}(N)$ and $b \in L$ such that

$$a_i \leq b \quad \forall i \in M, \qquad a_i \leq b' \quad \forall i \in \omega \backslash M$$

L is called a SCP-OML iff it has the *Subsequential Completeness Property*: For every orthogonal sequence $(a_i)_{i\in\omega} \subseteq L$ there exists $M \in \mathcal{I}(\omega)$ such that $\bigvee_{i\in M} a_i$ exists in L. If L is an OA and Q is a sub-OML of L, then Q is called a SIP-sub-OML (resp., SCP-sub-OML) if Q considered as an OML is a SIP-OML (resp., SCP-OML).

(ii) An OA L is called a WSIP-OA (resp., WSCP-OA) iff it satisfies the *Weak Subsequential Interpolation* (resp., *Completeness*) *Property*: For every sequence $(a_i)_{i\in\omega} \in J(L)$, there exist a subsequence $(a_{i_k})_{k\in\omega}$ and a SIP-sub-OML (resp., SCP-sub-OML) Q of L that contains $(a_{i_k})_{k\in\omega}$.

(iii) An OA L is called a σ-*orthoalgebra* (Habil, 1994*a*) if for every countable $X \in J(L)$ we have

$$\oplus X := \bigvee_{F\in\mathcal{F}(XC)} \oplus F$$

exists in L.

Remark 2.4. (1) Every SIP-OML is also a WSIP-OA, but not conversely. For example, the OA of Example 2.13 of Foulis *et al.* (1992), usually referred to as the Wright triangle, is a WSIP-OA that is not even an OMP.

(2) For an OA L, WSCP $\Rightarrow$ WSIP, but not conversely, as Freniche's example (Freniche, 1984, Theorem 7) shows.

(3) Every σ-orthoalgebra is a WSCP-OA (and, hence, a WSIP-OA). Indeed, this follows immediately from Theorem 3.11 of Habil (1994*a*) and Definition 2.3. However, the converse need not be true. In fact, Example 3.19 of Habil (1994*b*) is an example of a WSCP-OA that is not a σ-orthoalgebra.

(4) It is not true, in general, that every Boolean subalgebra of a SIP-OML has SIP. For instance, the OML $\mathcal{P}(\omega)$ has SIP, while the Boolean subalgebra $\mathbb{B}(\omega)$ that consists of all finite or cofinite subsets of ω does not.

3. PRELIMINARY LEMMAS

Definition 3.1. Let (S, d) be a pseudometric space. A *sequence* $(s_i)_{i\in\omega} \subseteq S$ is said to *converge* to s in (S, d) and we write $\lim_{i\to\infty} s_i = s$ if, given $\epsilon > 0$, $\exists i_0 = i_0(d, \epsilon) \in \omega$ such that $\forall i \geq i_0$ we have $d(s_i, s) < \epsilon$. An *infinite series* $\Sigma_{i\in\omega} s_i$, $(s_i)_{i\in\omega} \subseteq S$, is said to *converge* (or be *summable*) to s in (S, d) and we write $\lim_{F\in\mathcal{F}(\omega)} \Sigma_{i\in F} s_i = s$ (or simply $\Sigma_{i\in\omega} s_i = s$) if the sequence of partial sums $(\Sigma_{i=0}^{n} s_i)_{n\in\omega}$ converges to s. This is equivalent to the following: Given $\epsilon > 0$, $\exists F_0 = F_0(d, \epsilon) \in \mathcal{F}(\omega)$ such that $\forall F \in \mathcal{F}(\omega)$ with $F_0 \subseteq F$ we have $d(\Sigma_{i\in F} s_i, s) < \epsilon$. We call *d semi-invariant* if $\forall s, t, v \in S$, we have $d(s + v, t + v) \leq d(s, t)$. Using the semi-invariance of d and the triangle inequality, one can easily see that d satisfies the following inequality $\forall s, t, v, w \in S$:

$$d(s + t, v + w) \leq d(s, v) + d(t, w). \tag{3.1}$$

Definition 3.2. A quadruple $(S, +, 0, \mathcal{U})$, where S is a set, $+$ is a binary operation on S, 0 is a distinguished element of S, and $\mathcal{U}$ is a *uniformity* on S, is said to be a *uniform semigroup* if the following axioms are satisfied:

(S1) The binary operation $+$ is associative and commutative.

(S2) $\forall x \in S, x + 0 = x$.

(S3) The function $(x, y) \mapsto x + y$: $S \times S \to S$ is uniformly continuous.

It is well known (Page, 1978) that the uniformity $\mathcal{U}$ can be generated by a set $\mathcal{D}$ of continuous pseudometrics d on S that are semi-invariant. A sequence $(s_i)_{i\in\omega} \subseteq S$ converges to s in $(S, \mathcal{D})$ if it converges to s in (S, d) $\forall d \in \mathcal{D}$. A series $\Sigma_{i\in\omega} s_i$ converges (or is summable) to s in $(S, \mathcal{D})$ if it converges to s in (S, d) $\forall d \in \mathcal{D}$.

Two typical examples of uniform semigroups are $\mathbb{R}$ and $[0, \infty]$ under the usual addition.

Definition 3.3. Let L be an orthoalgebra and let S be a Hausdorff uniform semigroup. A function μ: $L \to S$ is called *additive* if:

(i) $\mu(0) = 0$;

(ii) and $\mu(\oplus_{i=1}^{n} a_i) = \Sigma_{i=1}^{n} \mu(a_i)$ for every finite jointly orthogonal subset $\{a_i: i = 1, \ldots, n\} \subseteq L$.

Since any pair of orthogonal elements in L is jointly orthogonal, then, as a consequence of (ii), we have:

(ii)′ $a, b \in L$ and $a \perp b \Rightarrow \mu(a \oplus b) = \mu(a) + \mu(b)$.

We shall write $a(L, S)$ for the set of all additive $\mu: L \to S$. A function $\mu: L \to S$ is called *s-bounded* if for every sequence $(a_i)_{i\in\omega} \in J(L)$ we have

$$\lim_{i\to\infty} \mu(a_i) = 0$$

and μ is called *order continuous* if for every decreasing sequence $(a_i)_{i\in\omega} \subseteq L$ such that $\bigwedge_{i\in\omega} a_i = 0$ we have

$$\lim_{i\to\infty} \mu(a_i) = 0$$

Let $sa(L, S)$ denote the set of all additive and s-bounded functions on L with values in S. A nonempty subset $M \subseteq sa(L, S)$ is called *uniformly s-bounded* if for every sequence $(a_i)_{i\in\omega} \in J(L)$ we have

$$\lim_{i\to\infty} \mu(a_i) = 0 \quad \text{uniformly in } \mu \in M$$

A nonempty subset $M \subseteq sa(L, S)$ is called *uniformly order continuous* if for every decreasing sequence $(a_i)_{i\in\omega} \subseteq L$ such that $\bigwedge_{i\in\omega} a_i = 0$, we have

$$\lim_{i\to\infty} \mu(a_i) = 0 \quad \text{uniformly in } \mu \in M$$

Note that if L is an OMP and $\mu: L \to S$ is a function, then our definitions of additivity, s-boundedness, and order continuity of μ coincide with the corresponding definitions given in the literature (De Lucia and Morales, 1988). This follows from the facts [see Corollary 3.5 and Lemma 4.1 of Habil (1994*a*)] that in an OMP every pairwise orthogonal (resp., decreasing) sequence is jointly orthogonal (resp., jointly compatible).

Henceforth, we assume that L is a WSIP-orthoalgebra, S is a Hausdorff uniform semigroup (HUS) with a fixed set $\mathfrak{D}$ of continuous pseudometrics that generate its uniformity, J(L) denotes the set of all jointly orthogonal subsets of L, a(L, S) denotes the set of all additive functions on L with values in S, and sa(L, S) denotes the set of all s-bounded members of a(L, S).

The following lemmas will lead to an important lemma which will be a key to proving the Brooks–Jewett theorem.

Lemma 3.4. $\mu \in a(L, S)$ is s-bounded iff

$$\begin{cases} \text{for every jointly orthogonal sequence } (a_i)_{i\in\omega} \subseteq L \text{ and} \\ \text{for every sub-OML } Q \supseteq (a_i)_{i\in\omega}, \text{ we have} \\ \lim_{i\to\infty} \mu(a_i \wedge^Q x) = 0 \text{ uniformly in } x \in Q \end{cases} \qquad (*)$$

Proof. ($\Leftarrow$): Assume (*) holds. Let $(a_i)_{i\in\omega} \in J(L)$ and let B be a block containing $(a_i)_{i\in\omega}$. Then, by (*),

$$\lim_{i\to\infty} \mu(a_i \wedge^B x) = 0 \quad \text{uniformly in } x \in B$$

Hence, since $x = 1 \in B$, we get $\lim_{i\to\infty} \mu(a_i) = 0$. Thus μ is s-bounded.

($\Rightarrow$): Suppose that μ is s-bounded. Let $(a_i)_{i\in\omega} \in J(L)$ and let Q be a sub-OML containing $(a_i)_{i\in\omega}$. Let $d \in \mathfrak{D}$ and $\epsilon > 0$ be given. We need to show that $\exists i = i(d, \epsilon) \in \omega$ such that

$$d(\mu(a_n \wedge^Q x), 0) < \epsilon \quad \forall n \geq i \quad \& \quad \forall x \in Q$$

Assume that this is not the case. Since $0 \in \omega$, $\exists n(0) \in \omega$, $n(0) > 0$, and $\exists x_0 \in Q$ such that

$$d(\mu(a_{n(0)} \wedge^Q x_0), 0) \geq \epsilon$$

Since $n(0) \in \omega$, we find $n(1) \in \omega$ with $n(1) > n(0)$ and $x_1 \in Q$ such that

$$d(\mu(a_{n(1)} \wedge^Q x_1), 0) \geq \epsilon$$

Continuing in this way, we obtain an increasing sequence $n(0) < n(1) < \ldots$ in ω, and a sequence $(x_k)_{k\in\omega} \subseteq Q$ such that

$$d(\mu(a_{n(k)} \wedge^Q x_k, 0) \geq \epsilon \qquad \forall k \in \omega \tag{3.2}$$

Set

$$c_k := a_{n(k)} \wedge^Q x_k \qquad (k \in \omega)$$

Since $(a_{n(k)})_{k\in\omega} \subseteq (a_i)_{i\in\omega}$ and since $(a_i)_{i\in\omega}$ is pairwise orthogonal, $(a_{n(k)})_{k\in\omega}$ is pairwise orthogonal. Since subelements of orthogonal elements are orthogonal, $(c_k)_{k\in\omega}$ is pairwise orthogonal. also, $c_k \in Q$ $\forall k$; so, as Q is a sub-OML, Corollary 3.5 of Habil (1994*a*) shows that $(c_k)_{k\in\omega} \in J(Q)$. Since it is clear that for any sub-OML Q of an OA L, $J(Q) \subseteq J(L)$, $(c_k)_{k\in\omega} \in J(L)$. Thus, by the s-boundedness of μ, we get $\lim_{k\to\infty} \mu(c_k) = 0$, which contradicts (3.2). ■

Lemma 3.5. Let $\mu \in sa(L, S)$. If $(a_i)_{i\in\omega} \in J(L)$ and if $(a_{i_k})_{k\in\omega} \subseteq (a_i)_{i\in\omega}$ and Q, a sub-OML containing $(a_{i_k})_{k\in\omega}$, are as provided by WSIP, then for every neighborhood V of 0 in S and for every $N \in \mathcal{I}(\omega)$, there exist $M \in$

$\mathcal{I}(N)$ and $b \in Q$ such that

$$a_{i_k} \leq b \quad \forall k \in M, \qquad a_{i_k} \leq b' \quad \forall k \in \omega\backslash M$$

and

$$\mu(b \wedge^Q x) \in V \qquad \forall x \in Q$$

Proof. Let V be a neighborhood of 0 in S and let $(N_j)_{j\in\omega}$ be a partition of ω by infinite sets. The hypothesis that Q is a SIP-sub-OML containing $(a_{i_k})_{k\in\omega}$ shows that for every $j \in \omega$, $\exists M_j \in \mathcal{I}(N_j)$ and $\exists b_j \in Q$ such that

$$a_{i_k} \leq b_j \quad \forall k \in M_j, \qquad a_{i_k} \leq b_j' \quad \forall k \in \omega\backslash M_j \tag{1,j}$$

Let $c_0 := b_0$ and $c_k := b_k \wedge^Q (\bigwedge_{j=1}^{Q\,k-1} b_j')$ for $k \in \omega\backslash\{0\}$. Evidently, the sequence $(c_k)_{k\in\omega} \subseteq Q$ and is pairwise orthogonal; hence it is jointly orthogonal since Q is a sub-OML. Thus, by Lemma 3.4,

$$\lim_{k\to\infty} \mu(c_k \wedge^Q x) = 0 \quad \text{uniformly in } x \in Q$$

Hence $\exists k_0 \in \omega$ such that $\mu(x \wedge^Q c_{k_0}) \in V\ \forall x \in Q$. By $(1, k_0)$, $k \in M_{k_0} \Rightarrow a_{i_k} \leq b_{k_0}$, and, since $M_{k_0} \cap M_j = \varnothing\ \forall j \neq k_0$, $k \in M_{k_0} \Rightarrow a_{i_k} \leq b_j'\ \forall j \neq k_0$. Therefore

$$a_{i_k} \leq b_{k_0} \wedge^Q (\bigwedge_{j=1}^{Q\,k_0-1} b_j') = c_{k_0} \qquad \forall k \in M_{k_0}$$

Also, by $(1, k_0)$,

$$k \in \omega\backslash M_{k_0} \quad \Rightarrow \quad a_{i_k} \leq b_{k_0}' \leq b_{k_0}' \vee^Q (\bigvee_{j=1}^{Q\,k_0-1} b_j) = c_{k_0}'$$

Thus the conclusion of the lemma is satisfied by $b := c_{k_0}$ and $M := M_{k_0}$. ■

Corollary 3.6. Let $\{\mu_0, \ldots, \mu_n\}$ be a subset of $sa(L, S)$. If $(a_i)_{i\in\omega} \in J(L)$ and if $(a_{i_k})_{k\in\omega} \subseteq (a_i)_{i\in\omega}$ and Q, a sub-OML containing $(a_{i_k})_{k\in\omega}$, are as provided by WSIP, then for every neighborhood V of 0 in S and for every $N \in \mathcal{I}(\omega)$, there exist $M \in \mathcal{I}(N)$ and $b \in Q$ such that

$$a_{i_k} \leq b \quad \forall k \in M, \qquad a_{i_k} \leq b' \quad \forall k \in \omega\backslash M$$

and

$$\mu_j(b \wedge^Q x) \in V \qquad \forall x \in Q \quad \& \quad \forall j \leq n$$

Proof. Let V be a neighborhood of 0 in S. Form $T = \overbrace{S \times \cdots \times S}^{n+1 \text{ times}}$. Since S is a HUS, so is T. Note that the function μ: $L \to T$ defined by

$$\mu(x) := (\mu_0(x), \ldots, \mu_n(x))$$

is additive and s-bounded, and $V^{(n+1)} = \overbrace{V \times \cdots \times V}^{n+1 \text{ times}}$ is a neighborhood of 0 in T. So if $(a_i)_{i\in\omega} \in J(L)$ and if $(a_{i_k})_{k\in\omega}$ and Q are as provided by WSIP, then, by Lemma 3.5, $\forall N \in \mathcal{I}(\omega)$ $\exists M \in \mathcal{I}(N)$ and $\exists b \in Q$ such that

$$a_{i_k} \leq b \quad \forall k \in M, \qquad a_{i_k} \leq b' \quad \forall k \in \omega \backslash M$$

and

$$\mu(b \wedge^Q x) \in V^{(n+1)} \qquad \forall x \in Q$$

Upon recalling that $\mu(b \wedge^Q x) = (\mu_0(b \wedge^Q x), \ldots, \mu_n(b \wedge^Q x))$, we see that this last inclusion means

$$\mu_j(b \wedge^Q x) \in V \qquad \forall x \in Q \quad \& \quad \forall j \leq n \quad \blacksquare$$

Lemma 3.7. Let $M \in \mathcal{I}(\omega)$ and let $(a_i)_{i\in\omega} \in J(L)$. If $(a_{i_k})_{k\in\omega} \subseteq (a_i)_{i\in\omega}$ and Q, a sub-OML containing $(a_{i_k})_{k\in\omega}$, are as provided by WSIP, then

$$\mathcal{G} := \{\Delta \in \mathcal{P}(M)\colon \exists b_\Delta \in Q \text{ with } a_{i_k} \leq b_\Delta\ \forall k \in \Delta,\ a_{i_k} \leq b'_\Delta\ \forall k \in \omega\backslash\Delta\}$$

is a Boolean subring of $\mathcal{P}(M)$ that has SCP, and $\{i\} \in \mathcal{G}$ $\forall i \in M$.

Proof. Let $(a_i)_{i\in\omega} \in J(L)$, and let $(a_{i_k})_{k\in\omega}$ and Q be as provided by WSIP. Let $\mathcal{G}$ be as defined above. We proceed in steps.

Step 1. $\mathcal{G}$ is a subring of $\mathcal{P}(M)$.

To see this, let $\Delta_1, \Delta_2 \in \mathcal{G}$. We must show that $\Delta_1 \cup \Delta_2, \Delta_1\backslash\Delta_2 \in \mathcal{G}$. Indeed, we have $\Delta_1 \cup \Delta_2, \Delta_1\backslash\Delta_2 \in \mathcal{P}(M)$ and $\exists b_1, b_2 \in Q$ such that

$$a_{i_k} \leq b_j \quad \forall k \in \Delta_j, \qquad a_{i_k} \leq b'_j \quad \forall k \in \omega\backslash\Delta_j \qquad (j = 1,2)$$

This implies that $\exists b := b_1 \vee^Q b_2 \in Q$ such that

$$a_{i_k} \leq b \quad \forall k \in \Delta_1 \cup \Delta_2, \qquad a_{i_k} \leq b' \quad \forall k \in \omega\backslash(\Delta_1 \cup \Delta_2)$$

Thus $\Delta_1 \cup \Delta_2 \in \mathcal{G}$. Also, $\exists p := b_1 \wedge^Q b'_2 \in Q$ such that

$$a_{i_k} \leq p \quad \forall k \in \Delta_1\backslash\Delta_2, \qquad a_{i_k} \leq p' \quad \forall k \in \omega\backslash(\Delta_1\backslash\Delta_2)$$

Thus $\Delta_1\backslash\Delta_2 \in \mathcal{G}$ and $\mathcal{G}$ is a Boolean subring of $\mathcal{P}(M)$.

Step 2. $\mathcal{G}$ has SCP.

To see this, let $(\Delta_r)_{r\in\omega}$ be a disjoint sequence in $\mathcal{G}$. Then, for every $r \in \omega$, $\exists$ a $b_r \in Q$ such that

$$a_{i_k} \leq b_r \quad \forall k \in \Delta_r, \qquad a_{i_k} \leq b'_r \quad \forall k \in \omega\backslash\Delta_r$$

Let $c_0 := b_0$ and $c_r := b_r \wedge^Q (\bigwedge_{j=1}^{Q\,r-1} b_j')$ for $r \in \omega\backslash\{0\}$. Evidently, $(c_r)_{r\in\omega}$ is pairwise orthogonal and

$$a_{i_k} \leq c_r \quad \forall k \in \Delta_r \quad \& \quad \forall r \in \omega \tag{3.3}$$

Consider the countable pairwise orthogonal subset

$$K := \{c_r: r \in \omega\} \cup \{a_{i_k}: k \in \omega\backslash \cup_{r\in\omega} \Delta_r\}$$

of Q. Since Q is a SIP-sub-OML, $\exists N \in \mathcal{I}(\omega)$ and $b \in Q$ such that

$$\begin{cases} c_r \leq b \quad \forall r \in N, \qquad c_r \leq b' \quad \forall r \in \omega\backslash N, \qquad \text{and} \\ a_{i_k} \leq b' \qquad \forall k \in \omega\backslash \cup_{r\in\omega} \Delta_r \end{cases} \tag{3.4}$$

We claim that $\cup_{r\in N} \Delta_r \in \mathcal{G}$. To see this, notice that (3.4) implies

$$a_{i_k} \leq b' \qquad \forall k \in \omega\backslash \cup_{r\in\omega} \Delta_r \tag{3.4$'$}$$

If $k \in \cup_{r\in\omega} \Delta_r$, then, as the Δ_r are disjoint, there exists a unique $q \in \omega$ such that $k \in \Delta_q$. Then, from (3.3), there exists a unique $q \in \omega$ such that

$$k \in \Delta_q \qquad \text{and} \qquad a_{i_k} \leq c_q \tag{3.5}$$

Now (3.4) and (3.5) imply that

$$\begin{aligned} a_{i_k} &\leq b \quad \text{if} \quad k \in \Delta_q \quad \text{with} \quad q \in N, \\ a_{i_k} &\leq b' \quad \text{if} \quad k \in \Delta_q \quad \text{with} \quad q \in \omega\backslash N \end{aligned}$$

that is,

$$a_{i_k} \leq b \quad \forall k \in \cup_{r\in N} \Delta_r, \qquad a_{i_k} \leq b' \quad \forall k \in \cup_{r\in \omega\backslash N} \Delta_r$$

and hence, by (3.4$'$), $a_{i_k} \leq b'$ $\forall k \in \omega\backslash \cup_{r\in N} \Delta_r$. Thus $\cup_{r\in N} \Delta_r \in \mathcal{G}$, as desired.

Step 3. $\{k\} \in \mathcal{G}$ $\forall k \in M$.

In fact, $k_0 \in M \Rightarrow \{k_0\} \in \mathcal{P}(M)$ and, as $(a_{i_k})_{k\in\omega}$ is pairwise orthogonal, we have

$$a_{i_k} \leq b := a_{i_{k_0}} \qquad \text{and} \qquad a_{i_k} \leq a_{i_{k_0}}' \qquad \forall k \in \omega\backslash\{k_0\}$$

Thus $\{k_0\} \in \mathcal{G}$ $\forall k_0 \in M$. ■

A HUS $(S, \mathcal{D})$ is *complete* if, for every $d \in \mathcal{D}$, (S, d) forms a complete pseudometric space.

Lemma 3.8. Let $\mu \in sa(L, S)$. If $(a_i)_{i\in\omega} \in J(L)$, then $(\sum_{i=0}^{k} \mu(a_i))_{k\in\omega}$ is a Cauchy sequence in S. Hence if S is a complete HUS, then $(\mu(a_i))_{i\in\omega}$ is summable in S.

Proof. Let $d \in \mathfrak{D}$. We need to show that for every $\epsilon > 0$, $\exists k_0 = k_0(\epsilon) \in \omega$ such that

$$d(s_m, s_j) < \epsilon \qquad \forall m > j \geq k_0$$

where $s_k := \Sigma_{i=0}^{k} \mu(a_i)\ \forall k \in \omega$. Suppose, contrariwise, that this does not hold. Then $\exists \epsilon > 0$ such that for each $k \in \omega$, $\exists j(k), m(k) \in \omega$ with $m(k) > j(k) > k$ and

$$d(s_{m(k)}, s_{j(k)}) > \epsilon$$

By the additivity of μ, we have $s_{m(k)} = s_{j(k)} + \mu(\oplus_{i=j(k)+1}^{m(k)} a_i)$. Hence, by the semi-invariance of d, we have

$$\begin{aligned}\epsilon &\leq d(s_{j(k)} + \mu(\oplus_{j(k)+1}^{m(k)} a_i), s_{j(k)} + 0)\\ &\leq d(\mu(\oplus_{j(k)+1}^{m(k)} a_i), 0) \qquad \forall k \in \omega\end{aligned}$$

Thus we can successively choose $k_0 < k_1 < \ldots$ in ω and $(j(k_0), m(k_0))$, $(j(k_1), m(k_1)), \ldots$ in $\omega \times \omega$ with $j(k_0) < m(k_0) < j(k_1) < m(k_1) < \ldots$ such that

$$d(\mu(\oplus_{i=j(k_r)}^{m(k_r)} a_i), 0) > \epsilon \qquad \forall r \in \omega$$

Set $c_r := \oplus_{i=j(k_r)+1}^{m(k_r)} a_i$ $(r \in \omega)$. Evidently, $(c_r)_{r\in\omega}$ is pairwise orthogonal; hence, the fact that $(a_i)_{i\in\omega} \in J(L)$ implies that $(c_r)_{r\in\omega} \in J(L)$ and therefore the preceding inequalities imply that

$$d(\mu(c_r), 0) > \epsilon \qquad \forall r \in \omega$$

This contradicts the s-boundedness of μ. ■

Lemma 3.9. Let S be a complete HUS, $(x_i)_{i\in\omega}$ be a summable sequence in S, and $(I_k)_{k\in K}$ be a partition of ω such that $(x_i)_{i\in I_k}$ is summable $\forall k \in K$. Then $(\Sigma_{i\in I_k} x_i)_{k\in K}$ is summable and $\Sigma_{k\in K}(\Sigma_{i\in I_k} x_i) = \Sigma_{i\in\omega} x_i$.

Proof. The proof is found in D'Andrea and De Lucia (1991), (3.1). ■

Lemma 3.10. Let S be a complete HUS and $\mu \in sa(L, S)$. For every $(a_i)_{i\in\omega} \in J(L)$, the function $\lambda: \mathcal{P}(\omega) \to S$ defined by

$$\lambda(\Delta) := \Sigma_{i\in\Delta}\, \mu(a_i) \qquad [\Delta \in \mathcal{P}(\omega)]$$

is σ-additive.

Proof. This follows immediately from Lemmas 3.8 and 3.9. ■

Lemma 3.11. Let (S, ρ) be a complete pseudometric semigroup, $(\mu_n)_{n\in\omega} \subseteq sa(L, S)$, and $(a_i)_{i\in\omega} \in J(L)$. If $(a_{i_k})_{k\in\omega} \subseteq (a_i)_{i\in\omega}$ and $Q \supseteq (a_{i_k})_{k\in\omega}$ are as provided by WSIP, then there exist a decreasing sequence $(M_i)_{i\in\omega} \subseteq \mathcal{I}((i_k)_{k\in\omega})$

and for every $i \in \omega$ an $e_i \in Q$ such that:

(1, i) $a_{i_k} \leq e_i\ \forall k \in M_i, \quad a_{i_k} \leq e_i'\ \forall k \in \omega \backslash M_i$.

(2, i) $M_i \subseteq M_{i-1} \backslash \{\min M_{i-1}\}$ (with $M_{-1} := \omega$).

(3, i) $\rho(\mu_p(x \wedge^Q e_i), 0) < 1/(i + 1)\ \forall p \leq i$ and $\forall x \in Q$.

(4, i) $e_{i+1} \leq e_i$.

Proof. Let $(a_i)_{i \in \omega} \in J(L)$ and let $(a_{i_k})_{k \in \omega}$ and Q be as provided by WSIP. We proceed by induction. Let $N = (i_k)_{k \in \omega} \backslash \{0\}$. By Lemma 3.5, $\exists M_0 \in \mathcal{I}(N)$ and $\exists e_0 \in Q$ such that

$$a_{i_k} \leq e_0 \quad \forall i_k \in M_0, \qquad a_{i_k} \leq e_0' \quad \forall i_k \in (i_k)_{k \in \omega} \backslash M_0$$

and

$$\rho(\mu_0(x \wedge^Q e_0), 0) < 1 \qquad \forall x \in Q$$

Let $n \in \omega$ and suppose that $e_0, e_1, \ldots, e_n$ and $M_0, M_1, \ldots, M_n$ have been constructed so that they satisfy (1, i), (2, i), and (3, i) for $i = 0, 1, \ldots, n$. Set $M := M_n \backslash \{\min M_n\}$. By Corollary 3.6, $\exists M_{n+1} \in \mathcal{I}(M)$ and $\exists b \in Q$ such that

$$a_{i_k} \leq b \quad \forall i_k \in M_{n+1}, \qquad a_{i_k} \leq b' \quad \forall i_k \in (i_k)_{k \in \omega} \backslash M_{k+1}$$

and

$$\rho(\mu_p(x \wedge^Q b), 0) < \frac{1}{n + 2} \qquad \forall x \in Q \quad \& \quad \forall p \leq n + 1 \tag{3.6}$$

Set $e_{n+1} := b \wedge^Q e_n$. Since, by $(1, n)$, $a_{i_k} \leq e_n\ \forall i_k \in M_n$ and $M_{n+1} \subseteq M_n$, we have

$$a_{i_k} \leq b \wedge^Q e_n = e_{n+1} \qquad \forall i_k \in M_{n+1}$$

and, since $a_{i_k} \leq b'\ \forall i_k \in (i_k)_{k \in \omega} \backslash M_{n+1}$, we have

$$a_{i_k} \leq b' \vee^Q e' = (b \wedge^Q e_n)' = e_{n+1}' \qquad \forall i_k \in (i_k)_{k \in \omega} \backslash M_{n+1}$$

Therefore $(1, n + 1)$ is satisfied. We also have $M_{n+1} \subseteq M_n \backslash \{\min M_n\}$, so $(2, n + 1)$ is satisfied. Finally, it follows from (3.6) that

$$\rho(\mu_p(x \wedge^Q e_n \wedge^Q b), 0) < \frac{1}{n + 2} \qquad \forall x \in Q \quad \& \quad \forall p \leq n + 1$$

That is, $(3, n + 1)$ holds and the induction is complete. ■

Now we are ready to establish the key lemma that will be used in proving the main result of this paper in the next section.

Lemma 3.12. Let (S, ρ) be a complete pseudometric semigroup and let $(\mu_n)_{n \in \omega} \subseteq sa(L, S)$. If $(a_n)_{n \in \omega} \in J(L)$ and $(a_{n_k})_{k \in \omega} \subseteq (a_n)_{n \in \omega}$ and Q are as provided by WSIP, then there exist a subsequence $(a_{m_i})_{i\in\omega}$ of $(a_{n_k})_{k\in\omega}$ and $N \in \mathcal{I}(\omega)$ such that:

(i) $\mathcal{G} := \{\Delta \in \mathcal{P}(N)\colon \exists b_\Delta \in Q$ with $a_{m_i} \leq b_\Delta\ \forall i \in \Delta,\ a_{m_i} \leq b'_\Delta\ \forall i \in \omega\backslash\Delta\}$ is a Boolean subring of $\mathcal{P}(N)$ with SCP (see Definition 2.3).

(ii) $\{i\} \in \mathcal{G}\ \forall i \in N$.

(iii) For any $n \in \omega$, the function $\lambda_n\colon \mathcal{G} \to S$ given by

$$\lambda_n(\Delta) = \Sigma_{i\in\Delta}\, \mu_n(a_{m_i})$$

is σ-additive and s-bounded.

(iv) For any $\Delta \in \mathcal{G}$, $\exists c_\Delta \in Q$ such that $\lambda_n(\Delta) = \mu_n(c_\Delta)\ \forall n \in \omega$.

(v) For any disjoint sequence $(\Delta_r)_{r\in\omega} \subseteq \mathcal{G}$, there corresponds a sequence $(c_r)_{r\in\omega} \in J(L)$ such that

$$\lambda_n(\Delta_r) = \mu_n(c_r) \qquad \forall n,\ r \in \omega$$

Proof. Let $(a_n)_{n\in\omega} \in J(L)$ and let $(a_{n_k})_{k\in\omega} \subseteq (a_n)_{n\in\omega}$ and Q be as provided by WSIP. Use Lemma 3.11 to pick a decreasing sequence $(e_i)_{i\in\omega} \subseteq Q$ and a decreasing sequence $(M_i)_{i\in\omega} \subseteq \mathcal{I}((n_k)_{k\in\omega})$ that satisfy (1, i), (2, i), and (3, i). Let $m_i := \min M_i\ \forall i \in \omega$. Then $(a_{m_i})_{i\in\omega}$ is a subsequence of $(a_{n_k})_{k\in\omega}$. Note that, for every $i \in \omega$,

$$[a_{m_i} < e_i \text{ (by } (1, i)),\ a_{m_i} < e'_{i+1} \text{ (by } (1, i+1) \text{ and } (2, i+1))]$$

implies that $a_{m_i} \leq e_i \wedge^Q e'_{i+1}$, and

$$[m_p \in M_{i+1}\ \forall p \geq i+1,\ m_p \in \omega\backslash M_i\ \forall p \leq i-1$$
$$(\text{since } M_p \subseteq M_{i+1}\ \forall p \geq i+1)]$$

implies that

$$a_{m_p} \leq e_{i+1} \qquad \forall p \geq i+1 \tag{3.7}$$

and

$$a_{m_p} \leq e'_i \qquad \forall p \leq i-1 \tag{3.8}$$

Now (3.7) and (3.8) imply that $a_{m_p} \leq e'_i \vee^Q e_{i+1}\ \forall p \in \omega\backslash\{i\}$ and therefore

$$a_{m_i} \leq e_i \wedge^Q e'_{i+1}, \qquad a_{m_j} \leq e'_i \vee^Q e_{i+1} \qquad \forall i \in \omega \quad \& \quad \forall j \in \omega\backslash\{i\} \tag{3.9}$$

Note that, by (3.9),

$$K := \{a_{m_i}\colon i \in \omega\} \cup \{e_i \wedge^Q e'_{i+1} \wedge^Q a'_{m_i}\colon i \in \omega\}$$

is a countable (pairwise) orthogonal subset of Q. So, as Q is an SIP-sub-OML, there exist $N \in \mathcal{I}(\omega)$ and $a \in Q$ such that

$$\begin{cases} a_{m_i} \leq a \quad \forall i \in N, \qquad a_{m_i} \leq a' \quad \forall i \in \omega\backslash N, \quad \text{and} \\ \qquad e_i \wedge^Q e'_{i+1} \wedge^Q a'_{m_i} \leq a' \qquad \forall i \in \omega \end{cases} \tag{3.10}$$

Let

$$\mathcal{G} := \{\Delta \in \mathcal{P}(N) \colon \exists b_\Delta \in Q \text{ with } a_{m_i} \leq b_\Delta \ \forall i \in \Delta,\ a_{m_i} \leq b'_\Delta \ \forall i \in \omega\backslash\Delta\}$$

Then, by Lemma 3.7, $\mathcal{G}$ is a Boolean subring of $\mathcal{P}(N)$ that has SCP and $\{i\} \in \mathcal{G}\ \forall i \in N$. Thus (i) and (ii) are proved.

To prove (iii), note first that, by Lemma 3.10, the function $\gamma_n \colon \mathcal{P}(\omega) \to S$ defined by

$$\gamma_n(\Delta) := \Sigma_{i\in\Delta}\, \mu_n(a_{m_i}) \qquad (n \in \omega)$$

is σ-additive. We claim that $\lambda_n := \gamma_n|_{\mathcal{G}}$ is s-bounded. To see this, let $(\Delta_r)_{r\in\omega}$ be a disjoint sequence in $\mathcal{G}$ and use the definition of $\mathcal{G}$ to pick a sequence $(b_r)_{r\in\omega} \subseteq Q$ such that

$$a_{m_i} \leq b_r \quad \forall i \in \Delta_r, \qquad a_{m_i} \leq b'_r \quad \forall i \in \omega\backslash\Delta_r \quad \& \quad \forall r \in \omega$$

Write

$$c_0 := b_0, \qquad c_r := b_r \wedge^Q (\wedge^{Q\,r-1}_{i=0}\, b'_i) \qquad \forall r \in \omega\backslash\{0\}$$

Evidently, $(c_r)_{r\in\omega}$ is pairwise orthogonal in Q; hence it is jointly orthogonal since Q is a sub-OML. Also,

$$a_{m_i} \leq c_r \qquad \forall i \in \Delta_r \quad \& \quad \forall r \in \omega \tag{3.11}$$

Since each μ_n is s-bounded, Lemma 3.4 implies that if V is a (closed) neighborhood of 0 in S, then $\exists r_0 \in \omega$ such that

$$\mu_n(x \wedge^Q c_r) \in V \qquad \forall r \geq r_0 \quad \& \quad \forall x \in Q$$

Thus, for every $r \geq r_0$, we have

$$\begin{aligned} \lambda_n(\Delta_r) &= \lim_{F\in\mathcal{F}(\Delta_r)} \Sigma_{i\in F}\, \mu_n(a_{m_i}) \\ &= \lim_{F\in\mathcal{F}(\Delta_r)} \mu_n(\oplus_{i\in F}\, a_{m_i}) \\ &\overset{(3.11)}{=} \lim_{F\in\mathcal{F}(\Delta_r)} \mu_n(c_r \wedge^Q (\vee^Q_{i\in F}\, a_{m_i})) \in V \end{aligned}$$

which shows that λ_n is s-bounded. This proves the claim and, hence, (iii).

To prove (iv), let $\Delta \in \mathcal{G}$. Then $\exists b_\Delta \in Q$ such that

$$a_{m_i} \leq b_\Delta \quad \forall i \in \Delta, \qquad a_{m_i} \leq b'_\Delta \quad \forall i \in \omega \backslash \Delta \tag{3.12}$$

We claim that $\lambda_n(\Delta) = \mu_n(e_0 \wedge^Q a \wedge^Q b_\Delta)$ $\forall n \in \omega$. To see this, note first that since $e_{i+1} \leq e_i$ $\forall i \in \omega$, we have

$$e_0 = (\bigvee\nolimits_{i=0}^{Q\,q-1} (e_i \wedge^Q e'_{i+1})) \vee e_q \qquad \forall q \in \omega$$

where $e_{-1} := 0$. By the OMI, (3.9) implies $e_i \wedge^Q e'_{i+1} = a_{m_i} \vee^Q ((e_i \wedge^Q e'_{i+1}) \wedge^Q a'_{m_i})$ $\forall i \in \omega$; so $\forall q \in \omega$, we have

$$\begin{aligned} e_0 &= (\bigvee\nolimits_{i=0}^{Q\,q-1} \mathrm{v}(e_i \wedge^Q e'_{i+1} \wedge^Q a'_{m_i})) \vee^Q (\bigvee\nolimits_{i=0}^{Q\,q-1} a_{m_i}) \vee^Q e_q \\ &= A_q \vee^Q B_q \end{aligned}$$

where

$$\begin{aligned} A_q &:= (\bigvee\nolimits_{i=0}^{Q\,q-1} (e_i \wedge^Q e'_{i+1} \wedge^Q a'_{m_i})) \vee^Q (\bigvee\nolimits_{i\in\{0,\ldots,q-1\}\cap(\omega\backslash\Delta)}^{Q} a_{m_i}) \\ B_q &:= (\bigvee\nolimits_{i\in\{0,\ldots,q-1\}\cap\Delta}^{Q} a_{m_i}) \vee^Q e_q \end{aligned}$$

Now note that $A_q \perp a \wedge^Q b_\Delta$ $\forall q \in \omega$. In fact, this follows from:

(a) $a_{m_i} \leq b'_\Delta$ $\forall i \in \omega\backslash\Delta$ implies that $a_{m_i} \leq a' \vee^Q b'_\Delta = (a \wedge^Q b_\Delta)'$ for every $i \in \{0, \ldots, q-1\} \cap (\omega\backslash\Delta)$;

(b) and $e_i \wedge^Q e'_{i+1} \wedge^Q a'_{m_i} \leq a'$ $\forall i \in \omega$ [by (3.8)].

Note also that $A_q \perp B_q$ $\forall q \in \omega$. In fact, this follows from:

(c) $\bigvee_{i=0}^{Q\,q} (e_i \wedge^Q e'_{i+1} \wedge^Q a'_{m_i}) \leq \bigwedge_{i\in\{0,\ldots,q-1\}\cap\Delta}^{Q} a'_{m_i}, e'_q$ $\forall q \in \omega$;

(d) and $\bigvee_{i\in\{0,\ldots,q-1\}\cap(\omega\backslash\Delta)}^{Q} a_{m_i} \leq \bigwedge_{i\in\{0,\ldots,q-1\}\cap\Delta}^{Q} a'_{m_i}, e'_q$ $\forall q \in \omega$;

which can be easily established using the facts that the set

$$\{a_{m_i}: i \in \omega\} \cup \{e_i \wedge^Q e'_{i+1} \wedge^Q a'_{m_i}: i \in \omega\}$$

is pairwise orthogonal and $(e_i)_{i\in\omega}$ is decreasing and by using (3.8). It follows that $A_q \perp (a \wedge^Q b_\Delta), B_q$ $\forall q \in \omega$; so the Foulis–Holland theorem (Kalmbach, 1983) and the fact that orthogonal elements are disjoint imply that

$$\begin{aligned} a \wedge^Q b_\Delta \wedge^Q e_0 &= a \wedge^Q b_\Delta \wedge^Q (A_q \vee^Q B_q) \\ &= (a \wedge^Q b_\Delta \wedge^Q A_q) \vee^Q (a \wedge^Q b_\Delta \wedge^Q B_q) \\ &= a \wedge^Q b_\Delta \wedge^Q ((\bigvee\nolimits_{i\in\{0,\ldots,q-1\}\cap\Delta}^{Q} a_{m_i}) \vee^Q e_q) \ \forall q \in \omega. \end{aligned}$$

Moreover, $c := \vee_{i\in\{0,\ldots,q-1\}\cap\Delta}^{Q} a_{m_i}$ is orthogonal to e_q [by (3.8)] and, by (3.10) and (3.12), we have

$$a_{m_i} \leq a \wedge^Q b_\Delta \qquad \forall i \in \{0, \ldots, q-1\} \cap \Delta \Rightarrow c \leq a \wedge^Q b_\Delta$$

It follows that $cC\,(a \wedge^Q b_\Delta), e_q$; hence the Foulis–Holland theorem implies that

$$a \wedge^Q b_\Delta \wedge^Q e_0 = [a \wedge^Q b_\Delta \wedge^Q (\bigvee^Q_{i\in\{0,\dots,q-1\}\cap\Delta} a_{m_i})] \vee^Q (a \wedge^Q b_\Delta \wedge^Q e_q)$$
$$= (\bigvee^Q_{i\in\{0,\dots,q-1\}\cap\Delta} a_{m_i}) \vee^Q (a \wedge^Q b_\Delta \wedge^Q e_a) \qquad \forall q \in \omega$$

Using the fact that subelements of orthogonal elements of an OA are orthogonal, it follows that $\bigvee^Q_{i\in\{0,\dots,q-1\}\cap\Delta} a_{m_i} \perp a \wedge^Q b_\Delta \wedge^Q e_q\ \forall q \in \omega$. Therefore, $\forall q, n \in \omega$, we have

$$\mu_n(a \wedge^Q b_\Delta \wedge^Q e_0) = \Sigma_{i<q, i\in\Delta}\, \mu_n(a_{m_i}) + \mu_n(a \wedge^Q b_\Delta \wedge^Q e_q) \tag{3.13}$$

Now, given $\epsilon > 0$ and $n \in \omega$, choose $q_0 \in \omega$, $q_0 > n$ such that (see the definition of γ_n)

$$\frac{1}{q_0} < \frac{\epsilon}{2} \qquad \text{and} \qquad \rho(\Sigma_{i<q,i\in\Delta}\, \mu_n(a_{m_i}), \gamma_n(\Delta)) < \frac{\epsilon}{2} \qquad \forall q \geq q_0$$

By $(3, q)$ of Lemma 3.11, we have

$$n < q \quad \Rightarrow \quad \rho(\mu_n(x \wedge^Q e_q), 0) < \frac{1}{q+1} < \frac{\epsilon}{2} \qquad \forall x \in Q \quad \& \quad \forall q \geq q_0$$

In particular,

$$\rho(\mu_n(a \wedge^Q b_\Delta \wedge^Q e_q), 0) < \frac{\epsilon}{2} \qquad \forall n < q \quad \& \quad \forall q \geq q_0$$

Thus, for every $q \geq q_0$, we have

$$\begin{aligned}\rho(\mu_n(a \wedge^Q b_\Delta \wedge^Q e_0), \gamma_n(\Delta)) &\overset{(3.13)}{=} \rho(\Sigma_{i<q,i\in\Delta}\, \mu_n(a_{m_i}) \\ &\qquad + \mu_n(a \wedge^Q b_\Delta \wedge^Q e_q), \gamma_n(\Delta)) \\ &\overset{(3.1)}{\leq} \rho(\Sigma_{i<q,i\in\Delta}\, \mu_n(a_{m_i}), \gamma_n(\Delta)) \\ &\qquad + \rho(\mu_n(a \wedge^Q b_\Delta \wedge^Q e_q), 0) \\ &< \epsilon.\end{aligned}$$

Since $\lambda_n = \gamma_n|_{\mathcal{G}}$, this proves the claim and, hence, (iv).

Finally, we prove (v). Let $(\Delta_r)_{r\in\omega}$ be a disjoint sequence in $\mathcal{G}$. For each $r \in \omega$, $\exists b_r := b_{\Delta_r}$ such that

$$a_{m_i} \leq b_r \quad \forall i \in \Delta_r, \qquad a_{m_i} \leq b_r' \quad \forall i \in \omega\backslash\Delta_r \tag{3.14}$$

Set $x_0 := b_0$, $x_r := b_r \wedge^Q (\bigwedge^{Q\,r-1}_{j=1} b_j')$ for $r \in \omega\backslash\{0\}$. Evidently, $(x_r)_{r\in\omega}$ is a pairwise orthogonal subset of Q; and, for every $r \in \omega$, we have

$$a_{m_k} \leq x_r \quad \forall k \in \Delta_r, \qquad a_{m_k} \leq x_r' \quad \forall k \in \omega \backslash \Delta_r$$

This shows that we may assume that the sequence $(b_r)_{r\in\omega}$ in (3.14) is pairwise orthogonal and we now make this assumption. Now set $c_r := e_0 \wedge^Q a \wedge^Q b_r$ $(r \in \omega)$. Then, since $(b_r)_{r\in\omega}$ is pairwise orthogonal, it follows that $(c_r)_{r\in\omega}$ is pairwise orthogonal in Q; hence it is jointly orthogonal since Q is a sub-OML. Moreover, the second claim above implies that

$$\lambda_n(\Delta_r) = \mu_n(c_r) \qquad \forall n, r \in \omega \quad \blacksquare$$

4. THE MAIN RESULTS

In this section, we shall state and prove the main results of this paper. In the sequel, we shall see that many of the results of this section will be immediate consequences of the following main result. Unless otherwise stated, the symbol $sa(L, S)$ continues to denote the set of all s-bounded and additive functions defined on an OA L with values in a HUS S with a fixed set $\mathfrak{D}$ of continuous pseudometrics.

Theorem 4.1 (Brooks–Jewett Theorem for WSIP-OAs). Let L be a WSIP-OA and $(\mu_n)_{n\in\omega} \subseteq sa(L, S)$. If

$$\lim_{n\to\infty} \mu_n(a) = \mu_0(a) \qquad \forall a \in L$$

Then $\{\mu_n: n \in \omega\}$ is uniformly s-bounded.

Proof. Suppose the contrary. Then we may assume, by passing to a subsequence if necessary, that $\exists d \in \mathfrak{D}$, $\exists \epsilon > 0$, and $\exists$ a sequence $(a_j)_{j\in\omega} \in J(L)$ such that

$$d(\mu_j(a_j), 0) > \epsilon \qquad \forall j \in \omega \tag{4.1}$$

Consider the following equivalence relation $\sim$ on S: $x \sim y$ iff $d(x, y) = 0$. Then under addition modulo $\sim$ the set $\tilde{S} := S/\sim$ of all equivalence classes becomes a HUS, and $\tilde{d}: \tilde{S} \times \tilde{S} \to \mathbb{R}$ defined by

$$\tilde{d}([x], [y]) := d(x, y)$$

is a semi-invariant metric on $\tilde{S}$. Hence $(\tilde{S}, \tilde{d})$ is a metric semigroup. Let π be the natural projection of S onto $\tilde{S}$. Clearly, π is continuous. Let (S_0, d_0) be the completion of $(\tilde{S}, \tilde{d})$ and let ι be the isometric imbedding of $(\tilde{S}, \tilde{d})$ into (S_0, d_0). Then ι is continuous and

$$d_0(\iota([x]), \iota([y]) = \tilde{d}([x], [y]) = d(x, y) \qquad \forall x, y \in S \tag{*}$$

Let $\nu_n := \iota \circ \pi \circ \mu_n \ \forall n \in \omega$. Then the s-boundedness of each μ_n and the continuity of $\iota \circ \pi$ imply that each ν_n is s-bounded. Thus $(\nu_n)_{n\in\omega} \subseteq sa(L, S_0)$. Moreover,

$$\lim_{n\to\infty} \nu_n(a) = \pi(\lim_{n\to\infty} \mu_n(a)) = \pi \circ \mu_0(a) = \nu_0(a) \qquad \forall a \in L \quad (4.2)$$

Let $(a_{n_k})_{k\in\omega} \subseteq (a_n)_{n\in\omega}$ and $Q \supseteq (a_{n_k})_{k\in\omega}$ be as provided by WSIP, and use Lemma 3.12 to pick a subsequence $(a_{m_i})_{i\in\omega}$ of $(a_{n_k})_{k\in\omega}$ and $N \in \mathcal{I}(\omega)$ such that the following hold:

(i) $\mathcal{G} = \{\Delta \in \mathcal{P}(N)\colon \exists b_\Delta \in Q \text{ with } a_{m_i} \leq b_\Delta\ \forall i \in \Delta,\ a_{m_i} \leq b'_\Delta\ \forall i \in \Delta\}$ is a Boolean subring of $\mathcal{P}(N)$ with SCP.

(ii) $\{i\} \in \mathcal{G}\ \forall i \in N$.

(iii) The functions $\lambda_n\colon \mathcal{G} \to S_0$ given by

$$\lambda_n(\Delta) := \Sigma_{i\in\Delta}\, \nu_n(a_{m_i}) \qquad (n \in \omega)$$

are σ-additive and s-bounded.

(iv) $\forall \Delta \in \mathcal{G}$, $\exists c_\Delta \in Q$ such that $\lambda_n(\Delta) = \nu_n(c_\Delta)\ \forall n \in \omega$.

Now (4.2) and (iv) imply that

$$\lim_{n\to\infty} \lambda_n(\Delta) = \lim_{n\to\infty} \nu_n(c_\Delta) = \nu_0(c_\Delta) = \lambda_0(\Delta) \qquad \forall \Delta \in \mathcal{G}$$

Using the fact (De Lucia and Morales, 1988, Theorem 2.1) that in a Boolean ring with SCP, the Brooks–Jewett theorem holds, we infer that $\{\lambda_n\colon n \in \omega\}$ is uniformly s-bounded. Hence, if $(i_k)_{k\in\omega}$ is a strictly increasing sequence in M, then, by (i), $(\{i_k\})_{k\in\omega}$ is a disjoint sequence in $\mathcal{G}$; so we have

$$\lim_{k\to\infty} \lambda_n(\{i_k\}) = 0 \quad \text{uniformly in } n \in \omega$$

Therefore, using (*), there exists $k_0 \in \omega$ such that

$$\begin{aligned} d(\mu_n(a_{m_{i_{k_0}}}), 0) &= d_0(\nu_n(a_{m_{i_{k_0}}}), 0) \\ &= d_0(\lambda_n(\{i_{k_0}\}), 0) \\ &< \epsilon \qquad \forall n \in \omega \end{aligned}$$

This contradicts (4.1). ■

The following result shows that if S in Theorem 4.1 is assumed to be a Hausdorff topological Abelian group, then it is not necessary to hypothesize the s-boundedness of μ_0. Some authors (Weber, 1986) refer to this type of result as a Vitali–Hahn–Saks theorem.

Theorem 4.2 (Brooks–Jewett Theorem for WSIP-OAs: Group-Valued Version). Let L be a WIPS-OA, S a Hausdorff topological Abelian group, and $(\mu_n)_{n\in\omega\setminus\{0\}} \subseteq sa(L, S)$. If

$$\lim_{n\to\infty} \mu_n(a) = \mu_0(a) \qquad \forall a \in L$$

then $\mu_0 \in sa(L, S)$ and $\{\mu_n: n \in \omega\}$ is uniformly s-bounded.

Proof. Repeat the proof of Theorem 4.1 and use the fact (De Lucia and Morales, 1988, Corollaries 2.2 and 2.3) that in a Boolean ring with SCP, the group-valued version of the Brooks–Jewett theorem holds to get the desired contradiction. ■

The following result gives a necessary and a sufficient condition for a sequence of s-bounded and finitely additive functions defined on a WSIP-OA to be uniformly s-bounded.

Theorem 4.3 (Cafiero's Theorem for WSIP-OAs). Let L be a WSIP-OA and $(\mu_n)_{n\in\omega} \subseteq sa(L, S)$. Then $\{\mu_n: n \in \omega\}$ is uniformly s-bounded if and only if for every sequence $(a_i)_{i\in\omega} \in J(L)$ and for every neighborhood V of 0 in S there exist $p, q \in \omega$ such that

$$\mu_n(a_p) \in V \qquad \forall n \geq q$$

Proof. ($\Rightarrow$): This part is obvious from the definition of uniform s-boundedness.

($\Leftarrow$): Suppose, contrariwise, that (μ_n) is not uniformly s-bounded. Then we may assume, by passing to a subsequence if necessary, that there exist a neighborhood U of 0 in S and a sequence $(a_i)_{i\in\omega} \in J(L)$ such that

$$\mu_i(a_i) \notin U \qquad \forall i \in \omega \tag{4.3}$$

We may assume that U is a d-neighborhood of 0 for some $d \in \mathfrak{D}$. Now to get the desired contradiction to (4.3), one basically repeats the proof of Theorem 4.1, and uses the fact that Cafiero's theorem holds for Boolean rings with SCP (Weber, 1986, Corollary 4.3 and §7). We omit the details. ■

Theorem 4.4 (Nikodym's Convergence Theorem for WSIP-OAs). Let L be a WSIP-OA and $(\mu_n)_{n\in\omega} \subseteq sa(L, S)$. If

$$\lim_{n\to\infty} \mu_n(a) \quad \text{exists in } S \text{ and equals } \mu_0(a) \quad \forall a \in L$$

and μ_n is order continuous for each $n \in \omega\backslash\{0\}$, then $(\mu_n)_{n\in\omega}$ is uniformly order continuous.

Proof. By Theorem 4.1, $(\mu_n)_{n\in\omega}$ is uniformly s-bounded. Suppose, contrariwise, that $(\mu_n)_{n\in\omega}$ is not uniformly order continuous. Then, by definition of uniform order continuity, we may assume, by passing to a subsequence if necessary, that there exist $d \in \mathfrak{D}$, $\epsilon > 0$, and a decreasing sequence $(a_i)_{i\in\omega}$

$\subseteq L$ with $\wedge_{i\in\omega} a_i = 0$ such that

$$d(\mu_i(a_i), 0) \geq \epsilon \qquad \forall i \in \omega \tag{4.4}$$

Choose $k_0 = 1$. Since μ_{k_0} is order continuous, $\exists k_1 \in \omega\backslash\{0\}$ such that

$$d(\mu_{k_0}(a_{k_1}), 0) < \frac{\epsilon}{2}$$

Since μ_{k_1} is order continuous, $\exists k_2 \in \omega\backslash\{0, 1, \ldots, k_1\}$ such that

$$d(\mu_{k_1}(a_{k_2}), 0) < \frac{\epsilon}{2}$$

Continuing inductively, we obtain a strictly increasing sequence $(k_j)_{j\in\omega}$ of natural numbers such that

$$d(\mu_{k_j}(a_{k_{j+1}}), 0) < \frac{\epsilon}{2} \qquad \forall j \in \omega \tag{4.5}$$

Since $a_{k_{j+1}} \leq a_{k_j}$ $\forall j \in \omega$, the OMI implies that

$$a_{k_j} = a_{k_{j+1}} \oplus (a_{k_{j+1}} \oplus a'_{k_j})' \qquad \forall j \in \omega$$

Set

$$b_j := (a_{k_{j+1}} \oplus a'_{k_j})' \qquad \forall j \in \omega$$

We claim that $(b_j)_{j\in\omega}$ is jointly orthogonal. To see this, let $i, j \in \omega$, $i \neq j$. We may assume that $i < j$. Then $i + 1 \leq j$ and we have

$$b_j = (a_{k_{j+1}} \oplus a'_{k_j})' \leq a_{k_j} \leq a_{k_{i+1}} \leq a_{k_{i+1}} \oplus a'_{k_i} = b'_i$$

Thus $(b_j)_{j\in\omega}$ is pairwise orthogonal. Since every chain in an OA is jointly compatible (Habil, 1994*a*, Lemma 4.1), it follows that there exists a block B of L such that $(a_i)_{i\in\omega} \subseteq B$. Hence $b_j = (a_{k_{j+1}} \oplus a'_{k_j})' \in B$ $\forall j \in \omega$ and therefore $(b_j)_{j\in\omega} \in J(L)$. This proves the claim.

Since $a_{k_j} = a_{k_{j+1}} \oplus b_j$ $\forall j \in \omega$, the additivity of μ_{k_j} implies that

$$\mu_{k_j}(a_{k_j}) = \mu_{k_j}(a_{k_{j+1}}) + \mu_{k_j}(b_j) \qquad \forall j \in \omega$$

so, by (3.2),

$$\begin{aligned} d(\mu_{k_j}(a_{k_j}), 0) &= d(\mu_{k_j}(b_j) + \mu_{k_j}(a_{k_{j+1}}), 0) \\ &\leq d(\mu_{k_j}(b_j), 0) + d(\mu_{k_j}(a_{k_{j+1}}), 0) \end{aligned}$$

which, by the use of (4.4) and (4.5), implies that

$$d(\mu_{k_j}(b_j), 0) \geq d(\mu_{k_j}(a_{k_j}), 0) - d(\mu_{k_j}(a_{k_{j+1}}), 0)$$
$$\geq \epsilon - \frac{\epsilon}{2} = \frac{\epsilon}{2} \qquad \forall j \in \omega$$

The last inequality shows that $(\mu_n)_{n\in\omega}$ cannot be uniformly s-bounded, which is a contradiction. ∎

The next result shows that if S in Theorem 4.4 is assumed to be a Hausdorff topological Abelian group, then it is not necessary to hypothesize the s-boundedness of μ_0.

Corollary 4.5. Let L be a WSIP-OA, S a Hausdorff topological Abelian group, and $(\mu_n)_{n\in\omega\setminus\{0\}} \subseteq sa(L, S)$. If

$$\lim_{n\to\infty} \mu_n(a) = \mu_0(a) \qquad \forall a \in L$$

and μ_n is order continuous for each $n \in \omega\setminus\{0\}$, then $(\mu_n)_{n\in\omega}$ is uniformly order continuous.

Proof. By Theorem 4.2, $\mu_0 \in sa(L, S)$. Thus $(\mu_n)_{n\in\omega} \subseteq sa(L, S)$ and therefore $(\mu_n)_{n\in\omega}$ is uniformly order continuous by Theorem 4.4. ∎

Remark 4.6. Note that Theorems 4.1–4.4 and Corollary 4.5 are also valid for WSCP-OAs since every WSCP-OA is also a WSIP-OA and they are also valid for SIP-OMLs (and, hence, for SCP-OMLs) since every SIP-OML is a WSIP-OA. Note further that these results are valid for σ-orthoalgebras since, by part 3 of Remark 2.4, every σ-orthoalgebra is a WSCP-OA. Thus Theorem 4.1 contains the results (5.1) and (6.1) of D'Andrea and De Lucia (1991) and the main theorem of Morales (1988) and Theorem 4.4 contains the corollary to the main theorem of Morales (1988).

The remaining part of this section is devoted to proving a Nikodym–Vitali–Hahn–Saks theorem for σ-additive functions defined on a σ-orthoalgebras. we start with the following definition.

Definition 4.7. Let L be a σ-orthoalgebra and S a HUS. A function μ: $L \to S$ is called *countably additive* (or *σ-additive*) iff for every $(a_i)_{i\in\omega} \in J(L)$, the infinite series

$$\mu(\bigoplus_{i\in\omega} a_i) = \Sigma_{i\in\omega}\, \mu(a_i)$$

converges in S. Note that every countably additive μ: $L \to S$ is, in particular, finitely additive and s-bounded whenever S is a group. In fact, let $(a_i)_{i\in\omega} \in J(L)$, $d \in \mathfrak{D}$, and $\epsilon > 0$ be given. By the countable additivity of μ, $\exists k_0 \in \omega$ such that $\forall k \geq k_0$, we have

$$d(\Sigma_{i=0}^{k}\, \mu(a_i), \mu(\bigoplus_{i\in\omega} a_i)) < \frac{\epsilon}{2}$$

Since S is a group, d is invariant. So this and the triangle inequality imply that $\forall k > k_0$, we have

$$\begin{aligned} d(\mu(a_k), 0) &= d(\mu(a_k) + \Sigma_{i=0}^{k-1} \mu(a_i), \Sigma_{i=0}^{k-1} \mu(a_i)) \\ &= d(\Sigma_{i=0}^{k} \mu(a_i), \Sigma_{i=0}^{k-1} \mu(a_i)) \\ &\leq d(\Sigma_{i=0}^{k} \mu(a_i), \mu(\bigoplus_{i\in\omega} a_i)) + d(\Sigma_{i=0}^{k-1} \mu(a_i), \mu(\bigoplus_{i\in\omega} a_i)) \\ &< \epsilon \end{aligned}$$

Thus μ is s-bounded.

From now on the symbol $ca(L, S)$ will denote the set of all countably additive functions defined on a σ-orthoalgebra L with values in a HUS S.

The following useful lemma is a generalization of Lemma 2.2, which appears in D'Andrea and De Lucia (1991) without proof, for orthoalgebras.

Lemma 4.8. Let L be a σ-orthoalgebra and $(\mu_n)_{n\in\omega} \subseteq ca(L, S)$ be uniformly s-bounded. If $(a_i)_{i\in\omega} \in J(L)$, then $(\mu_n(a_i))_{i\in\omega}$ is summable uniformly in $n \in \omega$.

Proof. We need to show that for every $d \in \mathfrak{D}$ and every $\epsilon > 0$, $\exists F^* = F^*(d, \epsilon)$ such that

$$d(\Sigma_{i\in F} \mu_n(a_i), \mu_n(\oplus_{i\in\omega} a_i)) < \epsilon$$
$$\forall F \in \mathcal{F}(\omega) \quad \text{with} \quad F \supseteq F^* \quad \text{and} \quad \forall n \in \omega$$

Suppose that this is false. Then $\exists d \in \mathfrak{D}$ and $\exists \epsilon > 0$ such that for every $F \in \mathcal{F}(\omega)$, $\exists K(F) \in \mathcal{F}(\omega)$ with $K(F) \supseteq F$ and

$$\sup_{n\in\omega} d(\Sigma_{i\in K(F)} \mu_n(a_i), \mu_n(\oplus_{i\in\omega} a_i)) > \epsilon$$

By the finite additivity of each μ_n and by the generalized associativity of $\oplus$ (Habil, 1994*a*, Theorem 3.16), we have

$$\mu_n(\oplus_{i\in\omega} a_i) = \mu_n(\oplus_{i\in K(F)} a_i) + \mu_n(\oplus_{i\in\omega\setminus K(F)} a_i)$$

So, by the semi-invariance of d, we obtain

$$\begin{aligned} \epsilon &< \sup_n d(\Sigma_{i\in K(F)} \mu_n(a_i) + 0, \mu_n(\oplus_{i\in K(F)} a_i) + \mu_n(\oplus_{i\in\omega\setminus K(F)} a_i)) \\ &\leq \sup_n d(0, \mu_n(\oplus_{i\in\omega\setminus K(F)} a_i)) \end{aligned}$$

Thus for each $F \in \mathcal{F}(\omega)$ we may (and do) choose an $n = n(F) \in \omega$ such that

$$d(\mu_{n(F)}(\oplus_{i\in\omega\setminus K(F)}a_i), 0) > \epsilon \tag{$*$}$$

Note that the countable additivity of each $\mu_{n(F)}$ implies that there exists $H(F) \in \mathcal{F}(\omega\setminus K(F))$ such that

$$d(\mu_{n(F)}(\oplus_{i\in H(F)}a_i), \mu_{n(F)}(\oplus_{i\in\omega\setminus K(F)}a_i)) < \frac{\epsilon}{2}$$

So the triangle inequality and $(*)$ yield that $\forall F \in \mathcal{F}(\omega)$, we have

$$\begin{aligned} d(\mu_{n(F)}(\oplus_{i\in H(F)}a_i), 0) &\geq d(\mu_{n(F)}(\oplus_{i\in\omega\setminus K(F)}a_i), 0) \\ &\qquad - d(\mu_{n(F)}(\oplus_{i\in H(F)}a_i), \mu_{n(F)}(\oplus_{i\in\omega\setminus K(F)}a_i) \\ &> \epsilon - \frac{\epsilon}{2} \geq \frac{\epsilon}{2} \end{aligned}$$

Thus we can successively choose sets $F_0, F_1, F_2, \ldots$ in $\mathcal{F}(\omega)$ and corresponding triples $(K(F_0), H(F_0), n(F_0))$, $(K(F_1), H(F_1), n(F_1))$, $(K(F_2), H(F_2), n(F_2))$, $\ldots$ such that

$$n(F_i) \in \omega \qquad \forall i \in \omega$$

$$F_i \subseteq K(F_i) \in \mathcal{F}(\omega) \qquad \forall i \in \omega$$

$$H(F_0) \in \mathcal{F}(\omega\setminus K(F_0)), \qquad H(F_i) \in \mathcal{F}(\omega\setminus(K(F_i) \cup \textstyle\bigcup_{j=0}^{i-1} H(F_j))) \qquad \forall i \geq 1$$

and

$$d(\mu_{n(F_j)}(\oplus_{i\in H(F_j)}a_i), 0) > \frac{\epsilon}{2} \qquad \forall j \in \omega$$

Now for $j = 0, 1, 2, \ldots$, set $c_j := \oplus_{i\in H(F_j)}a_i$. Since $(a_i)_{i\in\omega} \in J(L)$ and $(H(F_j))_{j\in\omega} \subseteq \mathcal{F}(\omega)$ is disjoint, we see that $(c_j)_{j\in\omega} \in J(L)$ and that $\forall j \in \omega$,

$$d(\mu_{n(F_j)}(c_j), 0) > \frac{\epsilon}{2}$$

This contradicts the uniform s-boundedness of $(\mu_n)_{n\in\omega}$. ∎

Definition 4.9. A subset $M \subseteq ca(L, S)$ is called *uniformly countably additive* iff for every $(a_i)_{i\in\omega} \in J(L)$, we have

$$\mu(\oplus_{i\in\omega} a_i) = \Sigma_{i\in\omega}\, \mu(a_i) \quad \text{uniformly in } \mu \in M$$

Now, we are ready to state and prove a Nikodym–Vitali–Hahn–Saks theorem for σ-orthoalgebras.

Theorem 4.10 (Nikodym–Vitali–Hahn–Saks Theorem for σ-Orthoalgebras). Let L be a σ-orthoalgebra and let S be a Hausdorff topological Abelian group. If $(\mu_n)_{n\in\omega\setminus\{0\}} \subseteq ca(L, S)$ is such that

$$\lim_{n\to\infty} \mu_n(a) = \mu_0(a) \qquad \forall a \in L$$

then $\mu_0 \in ca(L, S)$ and $(\mu_n)_{n\in\omega}$ is uniformly countably additive.

Proof. Let $(\mu_n)_{n\in\omega\setminus\{0\}}$ be as above. Since S is a group, the remark that follows Definition 4.7 shows that $ca(L, S) \subseteq sa(L, S)$. Let $(a_i)_{i\in\omega} \in J(L)$. Since L is a σ-orthoalgebra, we may write $X := \{a_i: i \in \omega\} \cup \{(\bigoplus_{i\in\omega} a_i)'\}$. Then, by Lemma 3.8 of Habil (1994*a*), $\bigoplus X = 1$. Hence, by Theorem 3.11 of Habil (1994*a*), $B := \{\bigoplus T: T \subseteq X\}$ is a complete Boolean subalgebra of L containing X. We now have $(\mu_n|_B)_{n\in\omega\setminus\{0\}} \subseteq sa(B, S)$ and $\lim_{n\to\infty} \mu_n(b) = \mu_0(b)$ $\forall b \in B$. So, by the Brooks–Jewett theorem for σ-orthoalgebras (see Theorem 4.2 and Remark 4.6), $(\mu_n|_B)_{n\in\omega}$ is uniformly s-bounded. As $(a_i)_{i\in\omega} \in J(B)$, it follows that

$$\lim_{i\to\infty} \mu_n(a_i) = 0 \quad \text{uniformly in } n \in \omega$$

This proves that $(\mu_n)_{n\in\omega}$ is uniformly s-bounded. Now, by Lemma 4.8, we infer that $(\mu_n)_{n\in\omega\setminus\{0\}}$ is uniformly countably additive.

It remains to show that $\mu_0 \in ca(L, S)$. Let $(a_i)_{i\in\omega} \in J(L)$, $d \in \mathfrak{D}$, and $\epsilon > 0$ be given. By the uniform countable additivity of $(\mu_n)_{n\in\omega\setminus\{0\}}$, $\exists F_0 \in \mathcal{F}(\omega)$ such that $\forall F \in \mathcal{F}(\omega)$ with $F_0 \subseteq F$ and $\forall n \in \omega\setminus\{0\}$, we have

$$d(\mu_n(\oplus_{i\in\omega} a_i), \Sigma_{i\in F}\mu_n(a_i)) < \epsilon \tag{*}$$

Hence, by continuity of d and of addition in S, we have $\forall F \in \mathcal{F}(\omega)$ with $F_0 \subseteq F$ that

$$\begin{aligned} d(\mu_0(\oplus_{i\in\omega} a_i), \Sigma_{i\in F}\mu_0(a_i)) &= d(\lim_{n\to\infty} \mu_n(\oplus_{i\in\omega} a_i), \Sigma_{i\in F} \lim_{n\to\infty} \mu_n(a_i)) \\ &= d(\lim_{n\to\infty} \mu_n(\oplus_{i\in\omega} a_i), \lim_{n\to\infty} \Sigma_{i\in F}\mu_n(a_i)) \\ &= \lim_{n\to\infty} d(\mu_n(\oplus_{i\in\omega} a_i), \Sigma_{i\in F} \mu_n(a_i)) \\ &\overset{(*)}{\leq} \epsilon \end{aligned}$$

Therefore $\mu_0 \in ca(L, S)$. ■

Remark 4.11. If P is an orthomodular poset, then it is not difficult to show (Habil, 1994*a*, Lemma 4.6) that P is σ-orthocomplete iff P is a σ-orthoalgebra. In this case,

$$\bigoplus_{i\in\omega} a_i = \bigvee_{i\in\omega} a_i \qquad \text{for all pairwise orthogonal } (a_i)_{i\in\omega} \subseteq P$$

Therefore, our definitions of countable additivity and uniform countable additivity coincide with the ones that are given in the literature (D'Andrea and De Lucia, 1991; Cook, 1978). Furthermore, since every σ-orthocomplete orthomodular poset is a σ-orthoalgebra, we conclude that Theorem 4.10 contains Proposition 6.4 of D'Andrea and De Lucia (1991) and Theorem 4 of Cook (1978) in the special case when the σ-orthoalgebra L is assumed to be a σ-orthocomplete orthomodular poset.

We conclude this section by giving an example which shows that there is no *Nikodym boundedness theorem* for σ-orthoalgebras (or even complete orthomodular lattices). Indeed, the next example provides a complete orthomodular lattice, a Hausdorff uniform topological group S, and an $M \subseteq sa(L, S)$ such that $M(a) = \{\mu(a)\colon \mu \in M\}$ is bounded in S [see Weber (1986) for the definition] for every $a \in L$, and yet $M(L) := \{\mu(a)\colon \mu \in M, a \in L\}$ is not bounded in S.

Example 4.12. Let B be the Boolean algebra of all finite or cofinite subsets of $\mathbb{N}$, and for every $n \in \mathbb{N}$ define $\delta_n\colon B \to \mathbb{R}$ by

$$\delta_n(E) = \begin{cases} 1 & \text{if} \quad n \in E \in \mathcal{F}(\mathbb{N}) \\ 0 & \text{if} \quad n \notin E \in \mathcal{F}(\mathbb{N}) \end{cases}$$

and

$$\delta_n(E) = -\delta_n(\mathbb{N}\setminus E) \qquad \text{if} \quad E \in c\mathcal{F}(\omega)$$

Clearly, $(\delta_n)_{n\in\mathbb{N}}$ is a sequence of bounded, countably additive measures on B. Let $L := \circ B_i$, the horizontal sum of B_i, $i \in \mathbb{N}$, where each B_i is an isomorphic copy of B [see Kalmbach (1983) for the definition of horizontal sum]. It is not difficult to check that L is a complete lattice. For each $j \in \mathbb{N}$, define functions $\mu_i^j\colon B_i \to \mathbb{R}$ by

$$\mu_i^j = i\delta_j \qquad \forall i \in \mathbb{N}$$

Then, for each $n \in \mathbb{N}$, define $\nu_n\colon L \to \mathbb{R}$ by

$$\nu_n(E) = \mu_i^n(E) \qquad \text{if} \quad E \in B_i$$

Clearly, $(\nu_n)_{n\in\mathbb{N}}$ is a sequence of additive and bounded functions on L. Moreover, it is not difficult to see that:

(i) $\lim_n(\sup_{E\in L}|\nu_n(E)|) = +\infty$.

(ii) For every disjoint sequence $(E_m)_{m\in}\mathbb{N}$ in L, there exists $i \in {}_{\mathbb{N}}$ such that $(E_m)_{m\in\mathbb{N}} \subseteq B_i$ and

$$\sup\{|\nu_n(E_m)|\colon n, m \in \mathbb{N}\} = i < \infty$$

Let 0, $1 \neq E \in L$. Then $E \in B_i$ for some $i \in \mathbb{N}$; and, hence, using (ii), we obtain

$$\sup\{|\nu_n(E)|: n \in \mathbb{N}\} = i < \infty$$

Thus, $(\nu_n)_{n\in\mathbb{N}}$ is pointwise bounded on L. However, (i) shows that $(\nu_n)_{n\in\mathbb{N}}$ is not uniformly bounded on L. We conclude that there is no Nikodym boundedness theorem for orthomodular lattices even under the assumption of completeness.

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